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Capillary waves on a vertical jet

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1. Introduction

Geer & Strikwerda (1983) devised a slender-jet theory for a jet falling vertically, with surface tension present. They solved the resulting problem numerically by extending a method they had developed previously (Geer & Strikwerda 1980). Their results showed that the cross-section of the jet decreased in area and oscillated in shape with distance along the jet. They compared the oscillations with Rayleigh's (1945) results for oscillations of a jet of constant circular cross-section. There was agreement qualitatively but not quantitatively.

We shall analyse the oscillations of a vertically falling jet on the basis of their theory in order to obtain a quantitative description of them which agrees with the numerical results. First we find the solution for a vertical jet with a circular cross-section. Then we determine its small-amplitude oscillations. Our analysis differs from Rayleigh's because our unperturbed jet is falling, so its velocity and radius are not constant.

2. Circular cross-section jet

For a circular jet, equations (2.3)–(2.5) of Geer & Strikwerda (1983) become, with $z' = 1+z$ and the prime omitted, and with w an appropriate Weber number,

$$\bar{\phi}_{rr} + r^{-1}\bar{\phi}_r = -\tfrac{1}{2}z^{-\frac{1}{2}} \quad (0 < r < \bar{s}(z), \quad z > 1), \tag{2.1}$$

$$\bar{\phi}_r = z^{\frac{1}{2}}\bar{s}_z \quad (r = \bar{s}(z)), \tag{2.2}$$

$$\bar{\phi}_r^2 + 2z^{\frac{1}{2}}\bar{\phi}_z = -\frac{1}{w\bar{s}} \quad (r = \bar{s}(z)). \tag{2.3}$$

The solution of these equations is

$$\bar{s}(z) = s_0 z^{-\frac{1}{4}}, \tag{2.4}$$

$$\bar{\phi}(r,z) = -\frac{r^2}{8z^{\frac{1}{2}}} - \frac{s_0^2}{32z} - \frac{2z^{\frac{3}{4}}}{3ws_0} + \phi_0. \tag{2.5}$$

Here s_0 is the jet radius at $z = 1$ and ϕ_0 is an irrelevant constant.

3. Waves

We now write

$$\phi(r,\theta,z) = \bar{\phi}(r,z) + \delta\phi'(r,\theta,z) + O(\delta^2), \quad s(\theta,z) = \bar{s}(z) + \delta s'(\theta,z) + O(\delta^2). \tag{3.1}$$

Here δ is a measure of the wave amplitude, which also indicates the deviation of the cross-section from a circle. By using (3.11) in equation (2.3)–(2.5) of Geer & Strikwerda (1983) and equating coefficients of δ, we obtain

$$\phi'_{rr}+r^{-1}\phi'_r+r^{-2}\phi'_{\theta\theta}=0 \quad (0 \leqslant r \leqslant \bar{s}(z)), \tag{3.2}$$

$$\phi'_r = z^{\frac{1}{2}}s'_z+\frac{s'}{4z^{\frac{1}{2}}} \quad (r=\bar{s}(z)), \tag{3.3}$$

$$2\bar{\phi}_r\phi'_r+2z^{\frac{1}{2}}\phi'_z=\frac{s'_{\theta\theta}}{w(\bar{s})^2}+\frac{s'}{w(\bar{s})^2}-\frac{\bar{s}s'}{8z^{\frac{3}{2}}}-\frac{\bar{s}s'}{8z} \quad (r=\bar{s}(z)). \tag{3.4}$$

A product solution of (3.2) is

$$\phi'(r,\theta,z)=U_2(z)\,r^n\cos n\theta. \tag{3.5}$$

Then we write the corresponding expression for s' as

$$s'(\theta,z)=U_1(z)\cos n\theta. \tag{3.6}$$

Substitution of (3.5) and (3.6) into (3.3) and (3.4) yields the following pair of equations for U_1 and U_2:

$$\frac{\mathrm{d}}{\mathrm{d}z}U_1=-\frac{1}{4z}U_1+n(\bar{s})^{n-1}z^{-\frac{1}{2}}U_2, \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}U_2=\left[\frac{1-n^2}{2w(\bar{s})^{n+2}z^{\frac{1}{2}}}-\frac{3}{16z^{\frac{3}{2}}(\bar{s})^{n-1}}\right]U_1+\frac{n}{4z}U_2. \tag{3.8}$$

To solve (3.7) and (3.8) we rewrite them as

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}U_1\\U_2\end{pmatrix}=\boldsymbol{A}(z)\begin{pmatrix}U_1\\U_2\end{pmatrix}. \tag{3.9}$$

Wavelike solutions of (3.9) are given by the method of Keller & Keller (1962), which involves the eigenvalues of the coefficient matrix $\boldsymbol{A}(z)$. They are

$$\lambda_{\pm}(z)=\frac{n-1}{8z}\pm\frac{\mathrm{i}}{2}\left[\frac{2(n^3-n)}{ws_0^3z^{\frac{1}{4}}}-\frac{(n-1)^2-8n}{16z^2}\right]^{\frac{1}{2}}\sim\frac{n-1}{8z}\pm\mathrm{i}\left(\frac{n^3-n}{2ws_0^3}\right)^{\frac{1}{2}}\frac{1}{z^{\frac{1}{8}}}. \tag{3.10}$$

The solution of (3.9) contains terms proportional to the exponentials

$$\exp\int_1^z\lambda_{\pm}(z)\,\mathrm{d}z\sim z^{\frac{1}{8}(n-1)}\exp\left[\pm\mathrm{i}\left(\frac{n^3-n}{2ws_0^3}\right)^{\frac{1}{2}}\tfrac{8}{7}(z^{\frac{7}{8}}-1)\right]=\exp[\pm\mathrm{i}b_2(z^{\frac{7}{8}}-1)\,w^{-\frac{1}{2}}] \tag{3.11}$$

where

$$b_2=\frac{8}{7}\left[\frac{n^3-n}{2s_0^3}\right]^{\frac{1}{2}}. \tag{3.12}$$

The exponent in (3.11) is the same as the argument of the sine in equation (5.4) of Geer & Strikwerda (1983). To compare (3.12) with the numerical values of b_2 in Geer & Strikwerda, we choose $n=2$, 3 and 4 for jets with elliptical, equilateral triangular and square cross-sections respectively. We choose s_0 so that the cross-sectional area πs_0^2 of the circular jet is equal to 4, the area of the orifice at $z=1$ in all three cases. This yields $s_0=2\pi^{-\frac{1}{2}}$. With this value, (3.12) yields the results shown in table 1 under 'analytical'. The numerical values from table 3 of Geer & Strikwerda (1983) are shown for comparison.

	Analytical	Numerical
Ellipse (2 to 1)	1.65	1.48
Triangle (equilateral)	3.30	2.95
Square	5.22	5.07

TABLE 1. Comparison of values of b_2

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